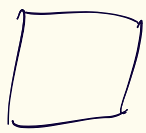
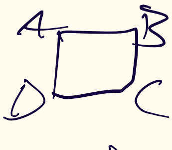
GROUP ACTIONS

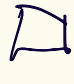
We defined groups as sets of symmetric transformations of objects, either sets or polyhedra. Let's expand upon that now.

$\boxed{\text{Eg}}$ Consider a square  and its group of transformations:
identity, 3 rotations, & 4 flips

This is called the <u>dihedral group</u> $D_{2\cdot 4}$

If we label the vertices  then

$$D_{2\cdot4} = \left\{ \begin{matrix} e, (AB)(CD), (AD)(BC), (AD), (BC), \\ (ABCD), (AC)(BD), (ADCB) \end{matrix} \right\}$$

This is an abstract algebraic object. In our description of $D_{2\cdot4}$ as transformations of  we say that $D_{2\cdot4}$ "acts" on the square, or, more particularly, the set of vertices.

Let's make this more precise: the "action" here is a map taking an element of $D_{2\cdot4}$ & an element of the set of vertices, and outputting another element of the set of vertices: $\alpha : D_{2\cdot4} \times X \to X$, $X = \{A, B, C, D\}$

$\boxed{\text{eg}}$ $\alpha((ABCD), B) = C$, $\alpha((AC)(BD), B) = D$

This idea is just a more abstract way of interpreting what a group is doing.

Groups are naturally defined by studying their actions on simplices (Subgroups of symmetric groups)

Recall one definition of symmetric gps: For any set X the group of permutations of X is a symmetry group, S_X.

Def^n An action of a group G on a set X is a homomorphism $\alpha : G \to S_X$

Def^n A <u>group action</u> of a group G on a set X is a map $: G \times X \to X$ s.t.

1. $\alpha(e, x) = x$
2. $\alpha(g, \alpha(h, x)) = \alpha(gh, x) \quad \forall g, h \in G$

HW: Prove the two def^{n}s are equivalent.

These are very broad definitions. For any group G & set X we can define a map $\alpha: G \times X \to X : (g, x) \mapsto x$, the trivial action.

Notation:
We will usually suppress the explicit writing of α.
Instead of $\alpha(g, x)$ we will write gx

$\boxed{Ex}$ Not all maps are actions. Let

$f: (\underset{\{0,1\}}{\mathbb{Z}_2}, +) \times \mathbb{Z} \to \mathbb{Z} : (g, z) \mapsto g \cdot z$

Then $f(\underset{\text{identity}}{0}, 2) = 0 \neq 2$ So ① fails.

$\boxed{Ex}$ S_n acts on $\underbrace{\{1, \dots, n\}}_{N}$ in the standard way:

$\alpha: S_n \times N \to N : (\sigma, i) \mapsto \sigma_i$

eg S_4 acts on $\{1, 2, 3, 4\}$, $\alpha((12)(34), 3) = 4$
$\alpha((243), 3) = 2$ etc.

$\boxed{Ex}$ Every group acts on itself trivially:

$\alpha: G \times G \to G : (g_1, g_2) = g_1 g_2$ $\quad \forall h, j, g \in G$

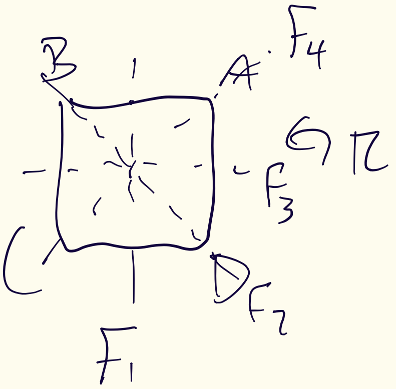
$\alpha(e, g) = eg = g$ & $\alpha(g, \alpha(h, j)) = \alpha(g, hj) = g(hj) = (gh)j$
$= \alpha(gh, j)$

$\boxed{\text{Ex}}$ Consider $D_{2\cdot 4}$, symm. trans. of a square.

flips & Rotations: $R_{90^\circ}=R$, $\langle R\rangle \subseteq D_{2\cdot 4}$, leaving 4 flips left. Consider any flip F. Since $|D_{2\cdot 4}:\langle R\rangle|=2$,

$F\langle R\rangle=\{F_1,F_2,F_3,F_4\}$ So $FR\in\{F_i\}$

So $(FR)^2=1$ $F^2=1$, $R^4=1$



$F_1 e = F_1 \begin{smallmatrix}B & A\\ C & D\end{smallmatrix} = \begin{smallmatrix}A & B\\ D & C\end{smallmatrix} = F_1$

$F_1 R = F_1 \begin{smallmatrix}A & D\\ B & C\end{smallmatrix} = \begin{smallmatrix}D & A\\ C & B\end{smallmatrix} = F_4$

$F_1 R^2 = F_1 \begin{smallmatrix}D & C\\ A & B\end{smallmatrix} = \begin{smallmatrix}C & D\\ B & A\end{smallmatrix} = F_3$

$F_1 R^3 = F_1 \begin{smallmatrix}C & B\\ D & A\end{smallmatrix} = \begin{smallmatrix}B & C\\ A & D\end{smallmatrix} = F_2$

Also, equivalently, Consider 2 flips $F_1\neq F_2$, $F_1F_2\neq R_{180}$

$\langle F_1,F_2\rangle=\{F_1,F_2,e,\underbrace{\overset{F_2F_1}{R}}_{\hookrightarrow\langle R\rangle},R^2,\overset{F_1F_2}{R^3},\overset{F_1R^2}{F_3},\overset{F_1R}{F_4}\}=D_{2\cdot 4}$

Relations between F_1 & F_2: $F_1^2=e$, $F_2^2=e$, $(F_1F_2)^4=e$

Now consider two coins

(H/T) (H/T)

1 2

with the following positions

1	2		2	1
H	H		H	H
H	T	&	H	T
T	H		T	H
T	T		T	T

$= X$

Consider $\alpha : D_{2\cdot 4} \times X \to X$ s.t. F_1 swaps coin 1

$\alpha\left(F_1, \underset{1}{H}\right) \mapsto \underset{1}{T}$ $\quad \alpha\left(F_1, \underset{1}{T}\right) \mapsto \underset{1}{H}$ $\quad$ & F_2 swaps coins 1&2

$\alpha\left(F_2, (1, 2)\right) \mapsto (2, 1)$ $\quad \alpha\left(F_2, (2, 1)\right) \mapsto (1, 2)$

What does $F_1 F_2$ do?

$$\alpha\left(F_1 F_2, \left(\underset{i \bmod 2}{H/T} \quad \underset{i+1 \bmod 2}{H/T}\right)\right) = \alpha\left(F_1, \left(\underset{i+1}{H/T} \quad \underset{i}{H/T}\right)\right) = \left(\underset{i+1}{T/H} \quad \underset{i}{H/T}\right)$$

$$\alpha\left(e, \left(\underset{i}{H/T} \quad \underset{i+1}{H/T}\right)\right) = \alpha\left(F_1^2, \left(\underset{i}{H/T} \quad \underset{i+1}{H/T}\right)\right) = \left(\underset{i}{H/T} \quad \underset{i+1}{H/T}\right)$$

This simple example draws a strong connection to combinatorics

ORBIT-STABILISER (AND ISO. THEOREMS)

Recall the first isomorphism theorem.

$\boxed{\text{Thm}}$ $\varphi: G \to H$ surjective hom. Then

$$\psi: G/_{\ker\varphi} \to H : g \cdot \ker(\varphi) \mapsto \varphi(g)$$

is an isomorphism. $\square$

This says we can study structure of a group by looking at homomorphisms in & out of it, i.e. ALL quotients correspond to some normal subgp, which is the kernel of some surj. hom. to another group, which is iso. to the quotient.

Group actions correspond to homomorphisms into symmetric gps, i.e. surjective hom. into some subgp. of symmetric gps. Recall Cayley's theorem: ALL Groups are subgps of symm. gps. So by studying gp actions we can learn structure about all gps!

In particular we relate gp actions to to cosets!

Orbit-Stabiliser

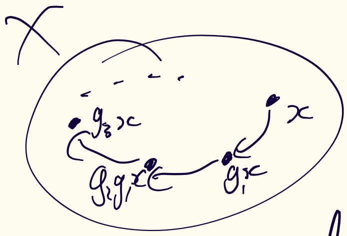
Groups "act" on sets. Let us study the iterative action on a single element.

Let G act on a set X.

let $x \in X$

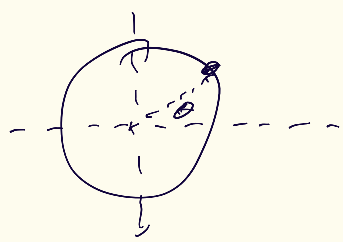
What does G do to this elt?

i.e. consider the set $\{g \cdot x \mid g \in G\}$



This is called the <u>orbit</u> of x. It is the set of elts in X that is preserved by G, it is the set of elts that G can send x to. ($\mathrm{Orb}(x)$)

Sometimes elts of G won't change x. Obviously the identity, but there may be others

<u>eg</u>) ① Let $G = \{z \in \mathbb{C} \mid |z| = 1\}$ = 

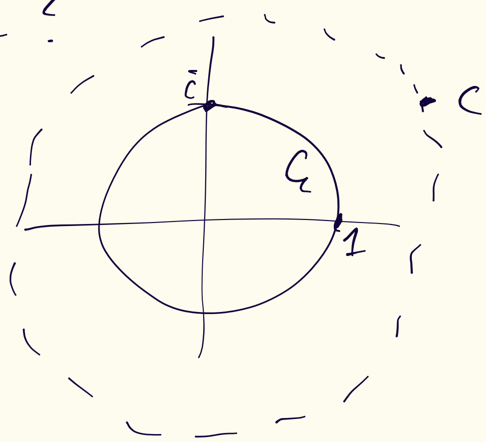
Points on circle with mult.

Consider the action on $\mathbb{C}$ by mult.

$z \in G \quad z = e^{i\theta}, \; c \in \mathbb{C}, \; c = \lambda e^{i\omega}$

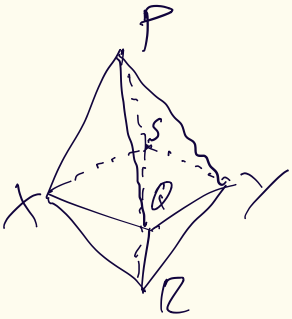
Then $z \cdot c = zc = \lambda e^{i(\theta + \omega)}$

So we are rotating c about the origin by θ. So what is the orbit of some point in $\mathbb{C}$?



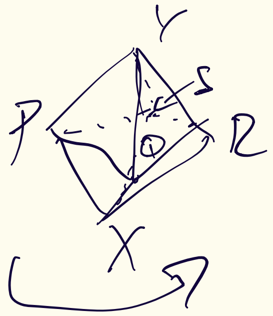
it's the circle about the origin that c belongs to!

(11) Consider a regular octahedron and the group of rotational symmetries, G.



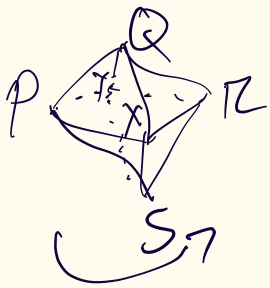
What is the orbit of P?

Rearranging:



P can go to Q, R, S by rotating around $\overline{YX}$ by $90°, 180°, 270°$

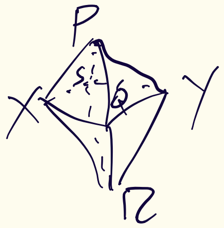
Similarly:



P can go to X & Y by rotation about $\overline{QS}$ by $90°$ & $270°$ respectively

The subset of elts of G that do not change x is called the stabiliser of x.

$$\mathrm{Stab}(x) = \{ g \in G \mid gx = x \}$$

Eg  What rotations fix P?
Any rotation about $\overline{PR}$ fixes P, there are 4 of them, $e, 90°, 180°, 270°$.
These form a group! ($\cong \mathbb{Z}_4 \leq$ Group of rotations)

Eg $G = \{z \in \mathbb{C} \mid |z| = 1\}$, acting on $\mathbb{C}$.
What fixes $c \in \mathbb{C}$? $z \in G$, $z = e^{i\theta}$, $c = \lambda e^{i\omega}$
$zc = \lambda e^{i(\theta+\omega)} = c = \lambda e^{i\omega}$ So z must be 1, i.e.
$\theta = 0$ $(\theta \in [0, 2\pi))$. So $\mathrm{Stab}(c) = \{1\}$. (Also a group)

In this example $\mathrm{Stab}(x) < G$, a subgp.
This is in fact always the case.

Lemma: Let G act on X, and $x \in X$. Then the stabiliser $\mathrm{Stab}(x)$ is a subgp of G, so $\mathrm{Stab}(x) < G$

Proof: Clearly $e \in \mathrm{Stab}(x) \subseteq G$. We only need to check that if $g_1, g_2 \in \mathrm{Stab}(x)$ then $g_1 g_2 \in \mathrm{Stab}(x)$ & $\forall g \in \mathrm{Stab}(x)$, $g^{-1} \in \mathrm{Stab}(x)$. $g_1 \cdot g_2\, x = g_1(g_2 x) = g_1 \cdot x = x$
So $g_1 g_2 \in \mathrm{Stab}(x)$ (if $gx = x$ then $g^{-1}x = g^{-1}(gx) = (g^{-1}g)x = x$)
$x = e \cdot x = g^{-1} \cdot \underbrace{gx}_{g \in \mathrm{Stab}(x)} = g^{-1} \cdot x$. So $g^{-1} \in \mathrm{Stab}(x)$. $\square$

We can look at the cosets of $\mathrm{Stab}(x)$ now.
Let's use the example again.

$\circlearrowright$ S_3 symmetry gp. So S_3 acts on $\{A,B,C\}$

$\mathrm{Stab}(A) = \{g \in S_3 \mid g \cdot A = A\} = \{e, F_A\}$ ← fix A

Cosets? $F_B \cdot \mathrm{Stab}(A) = \{F_B, R_2\}$ ← send A to C

$F_C \cdot \mathrm{Stab}(A) = \{F_C, R_1\}$ ← send A to B

$(R_2 = R_{240°})$
$(R_1 = R_{120°})$

3 cosets.

What is the orbit of A?

$\mathrm{Orb}(A) = \{gA \mid g \in S_3\} = \{A, B, C\}$, 3 elts.

In fact there is a natural bijection

$\{\text{left cosets of } \mathrm{Stab}(A)\} \to \mathrm{Orb}(A) : g\,\mathrm{Stab}(A) \mapsto gA$

any element from one coset sends A to the same place!

This is not a coincidence!

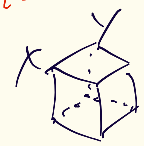
Theorem: (Orbit-Stabiliser theorem)

Let G be a group acting on X and $x \in X$.
Then there is a bijection

$\{\text{left cosets of } \mathrm{Stab}(x)\} \to \mathrm{Orb}(x) : g\,\mathrm{Stab}(x) \mapsto gx.$

Corollary: If G is finite & acts on X then $|G| = |Stab(x)| \cdot |\underbrace{Orb(x)}_{\#\text{ of cosets of } |Stab(x)|}| \quad \forall x \in X$. $\square$

Example: Consider a cube



What is the group G of symmetric transformations?

Note that no physical transformation swaps X & Y and no other points, so $G \neq S_{8 \leftarrow 8 \text{ vertices}}$

Consider the orbit of a face F. Where can F be sent?

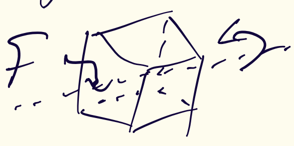
To all possible faces (compose rotations), so

$|Orb(F)| = 6$

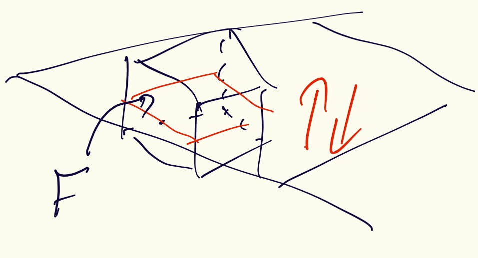
$$|G| = |Stab(F)| \cdot |Orb(F)| = 6 \cdot |Stab(F)|$$

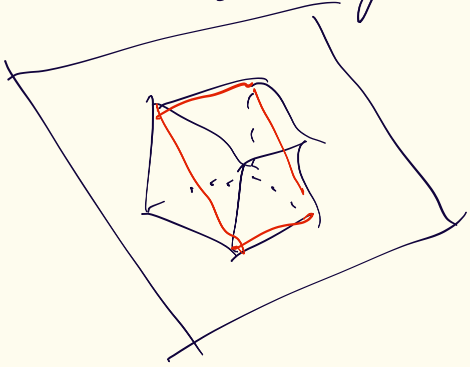
What transformations preserve F?

The four rotations about the axis through F



Also 4 flips across planes through F





Note that $Stab(F) \cong D_{2 \cdot 4}$

So $|Stab(F)| = 8$, So $|G| = 6 \cdot 8 = 48$

Remark:

One can show that G is the semi-direct product of the group of rotations (S_4) & $\mathbb{Z}_2$, which represents reflections, i.e. every symmetry is a combination of some rotation & a reflection.
We won't spend any more time on semi-direct products.

Now we will prove the orbit-stabiliser theorem.

Proof of orbit-stabiliser:

G acting on X, $x \in X$.
Then $\varphi : G/Stab(x) \to Orb(x) : g\,Stab(x) \mapsto gx$

Well defined? if $g \in h\,Stab(x)$ then $gx = hx$
So $\varphi(g) = gx = hx = \varphi(h)$

Onto: Arbitary elt in $Orb(x)$ is gx for some $g \in G$. then $\varphi(g\,Stab(x)) = gx$.

1-1: $g, h \in G$. Then

$$\begin{aligned} &\varphi(g) = \varphi(h) \\ \Leftrightarrow\ & gx = hx \\ \Leftrightarrow\ & h^{-1}gx = x \\ \Leftrightarrow\ & h^{-1}g \in Stab(x) \\ \Leftrightarrow\ & g \in h\,Stab(x) \\ \Leftrightarrow\ & g\,Stab(x) = h\,Stab(x) \end{aligned}$$

$\square$